Photon and Electron Spins†

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Received: July 2, 2009; Revised Manuscript Received: October 14, 2009

It is easy to draw an intuitive parallel between the classical free electromagnetic field and its corresponding quantum, the photon—a spin-1 object. The situation with a massive particle such as an electron is less clear, as a real-world analog of the classical field whose quantum is the massive particle is not available. It is concluded that the fermion *particle* perspective provides the best avenue for an intuitive grasp of the spin of an elementary fermion.

I. Introduction

The field of chemical dynamics is a mature one, in which it has become the norm for results from detailed experimental studies to be compared to results from high-level theoretical calculations. Advances in both experiment and theory have been motivated by, and are congruent with, ambitious yet sensible goals: deep qualitative understanding of underlying factors that govern chemical transformation, and accurate predictions of even the most nuanced experimental measurements.

Scientists in this area—experimentalists and theoreticians alike—have been steadfast in their commitment to improve insight and intuition at progressively higher levels of detail. For example, nowadays the chemical dynamics community deals routinely with weak spin—orbit interactions as a system moves from its entrance channels through its exit channels. Moreover, it turns out that such interactions, despite their modest energies, can be important, even decisive, insofar as influencing reaction pathways is concerned.

Accord between experiment and theory is sought, but often not easily achieved, at the highest levels of detail. Witness, for example, the reactions of $^2P_{1/2}$ and $^2P_{3/2}$ halogen atoms with simple molecules. Even with light nuclei, and therefore small spin—orbit splittings, the reactivities of these species can differ qualitatively.[1−4] With heavy nuclei, relativistic effects such as spin—orbit interaction can be so important as to alter potential surfaces in ways that have no counterparts in systems comprised of light nuclei.[5,6] The path to success has been arduous, but persistent efforts that have spanned decades are now paying off with remarkable agreement between first-principles theory and exquisite experiments.[1] For example, the pioneering experimental and theoretical work of Aquilanti and co-workers[7−12] set the stage for a generation of studies of reactive and inelastic scattering of open-shell species, including a number of important effects attributed to spin—orbit interaction.

This article is also about spin, namely, the intrinsic spin of an elementary entity such as a photon or an electron. It has nothing to do per se with comparisons between theoretical calculations and experimental results, nor does it address spin's dynamical role in reactive and inelastic scattering. It is about spin itself. The aim is to provide a means whereby intrinsic spin can be understood at an intuitive level. Particular attention is paid to the photon and spin-$^1/_2$ particles such as electrons, as these species are of paramount importance, not only in chemical dynamics, but in all of physical science.

The intrinsic spin of an elementary particle is a subject rife with subtlety. As used here, the term *elementary particle* means that the particle is not made of other things. For example, an electron is an elementary particle, but a proton is not, because it is made of quarks bound by gluons. In this sense, the term elementary particle can include things that have no mass, such as photons.

Pieter Zeeman discovered spin in the late 1890s, well before there was a theory of quantum mechanics, and to this day spin is integral to the most mathematically rigorous theories of the physical world. We are deft at manipulating spins, as well as dealing with their many applications and inventing new ones. At the same time, chemical dynamicists, for the most part, sidestep obvious but vexing questions: What is spin? What is the best way to visualize it? Is it quantum mechanical? After all, intrinsic spins of elementary things such as the electron and the photon have just two quantum states. Are there classical analogs? Is spin relativistic? After all, *s*pin-$^1/_2$ seems to pop out of the Dirac equation. These are hard questions worthy of attention.

Though such questions have been pondered for years, they go largely unanswered. For example, Ohanian suggested an intuitive picture for spin.[13] With classical electromagnetic and Dirac fields as examples, it was concluded that spin could be interpreted as a circulation of momentum in the classical wave fields whose quantizations yield a photon and a massive spin-$^1/_2$ particle, respectively. Comparison between a classical electromagnetic field and a photon is sensible but requires a more careful look at the field's spin density. This is facilitated by the application of Noether's theorem. As discussed below, the case of a massive spinor (Dirac) field is subtler, as no real-world analog of the classical field of a massive particle is known.

In what follows, an overview is given of Noether's theorem as it applies to classical fields. This is needed for the subsequent discussion. Noether's theorem has been around for almost a century and detailed accounts can be found elsewhere. It is used widely in theoretical physics, much less in chemical dynamics. Its use here is restricted to massive spinor and massless vector (electromagnetic) fields. Its application to the electromagnetic field reveals, without ado, a spin density that yields photon spin straightaway. The issue of canonical versus symmetrized tensors, each of which can be used with Noether's theorem, is discussed. Without doubt, the canonical tensor provides the greatest

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transparency and insight. The Dirac field is treated similarly, again supporting the use of the canonical tensor rather than a symmetrized one. Insight into the electromagnetism case is more straightforward than in the spin-$1/2$ case. Nonetheless, one sees with minimal math how the spin density is manifest with the classical Dirac field.

Finally, the most important issues are reached. Is the classical Dirac field useful in the quest for an intuitive understanding of the spin of an elementary particle such as an electron? It is argued that this is not a good way to visualize the spin of a massive particle. Unlike electromagnetism, where classical fields can serve as excellent descriptors of nature, massive fields have no real-world classical analogs. An alternate perspective, in which the particle is assigned the privileged role, is recommended. The fermion *particle* obeys the exclusion rule, and this endows it with what we call spin.

A word about language and symbols is in order. With a few exceptions (which will be identified explicitly), superscripts and subscripts denote contravariant and covariant components, respectively. Standard relativistic 4-vector notation is used throughout: implied summation for repeated indices (one upper, one lower); Greek letters for 4-vectors (e.g., x^μ); Roman letters and arrows for 3-vectors (e.g., x^i, $\vec{x}$, $\vec{\nabla}$); covariant and contravariant derivatives: $\partial_\mu \equiv \partial/\partial x^\mu$ and $\partial^\mu \equiv \partial/\partial x_\mu$, respectively. The Minkowski metric $(+ - - -)$ is used, for example to raise and lower indices, e.g., $F_{\varepsilon\rho} = g_{\varepsilon\mu}F^{\mu\nu}g_{\nu\rho}$. Except where necessary for clarity, the constants $\hbar$, c, μ_0, and ε_0 are each set equal to 1.

II. Background

A theorem due to Emmy Noether provides a straightforward means of examining the intrinsic spins of classical fields. It deals with conserved quantities that are identified through invariance of the action under continuous symmetry transformations. With the end points of the variations held fixed, covariance of the equations of motion is assured through invariance of the Lagrangian density, whose effect on the action is unchanged by the addition of a 4-divergence.

Emmy Noether was a prolific mathematician with important theorems to her credit. The one under consideration here states that if a system has a continuous symmetry there exists a related quantity whose value is conserved, i.e., it is constant in time. The conserved quantity is called the Noether charge and its associated flow is called the Noether current. We are familiar with electric charge, such as that of an electron. Noether charge is a generalization of this.

Though Noether's theorem is a cornerstone of theoretical physics, most mathematicians have not heard of it, as Emmy Noether's greatest contributions in mathematics were in the areas of abstract algebra, group theory, ring theory, group representations, and number theory. She moved from Erlangen to Göttingen in 1916 at the invitation of David Hilbert and Max Klein. The idea was that she would participate with Hilbert, Klein, and Einstein in the resolution of daunting mathematical issues that had arisen in the theory of general relativity. And participate she did—straightening out the mathematics of what is one of (if not *the*) greatest physics theories of the twentieth century. It is interesting that, despite its widespread use in quantum physics, Noether's theorem preceded quantum mechanics by a decade, and it was developed for entirely different purposes.

The present paper is concerned with spins of classical fields rather than aspects of Noether's theorem per se. Thus, equations relevant to Noether's theorem will be presented with brief descriptions and justifications, and we will proceed from there. Detailed derivations are available elsewhere. The one in Schwabl is excellent and will be referred to a couple times.[14] Soper provides truly outstanding insights,[15] though the math is less traditional [e.g., the metric $(- + + +)$ is used rather than $(+ - - -)$], and it is often hard to follow. In this paper, the simplest version of Noether's theorem that is applicable to the fields of interest is used.

To begin, consider the variation of a Lagrangian density L that depends on the fields and their first derivatives, but not explicitly on spacetime: $L(\phi_r, \partial_\mu \phi_r)$. The index r labels the fields. When dealing with a real scalar field, r is irrelevant so it is dropped. For a complex scalar field, r has two labels (for ψ and $\psi^\dagger$). For an electromagnetic field, there are 4 components, so r is assigned the labels 0, 1, 2, and 3. For a Dirac spinor there are 8 components (4 for the spinor and 4 for the adjoint spinor). And so on for other fields and combinations of fields. In principle there is no limit to the number of labels that can be assigned to r.

For an inhomogeneous Lorentz transformation, the Lagrangian density (hereafter referred to simply as the Lagrangian) responds to variations, induced by a change of reference frame, of the spacetime points and the fields. Because the symmetry is continuous, infinitesimal displacements define the transformation properties:[16]

$$\delta x_\nu = \Delta\omega_{\nu\sigma}x^\sigma + \delta_\nu \quad (1)$$

$$\Delta\phi_r = \frac{1}{2}\Delta\omega_{\nu\sigma}S^{\nu\sigma}_{rs}\phi_s \quad (2)$$

The term δ_ν is an infinitesimal translational displacement, while the term $\Delta\omega_{\nu\sigma}$ is a different kind of infinitesimal displacement. For example, $\Delta\omega_{01}$ corresponds to an infinitesimal boost, while $\Delta\omega_{12}$ corresponds to an infinitesimal rotation. Note that $\Delta\omega_{\nu\sigma}$ is antisymmetric with respect to interchange of its indices: $\Delta\omega_{\nu\sigma} = -\Delta\omega_{\sigma\nu}$. There are 6 independent $\Delta\omega_{\nu\sigma}$'s: 3 boosts ($\Delta\omega_{0i}$) and 3 rotations ($\Delta\omega_{ij}$). The term $S^{\nu\sigma}_{rs}$ depends on the nature of the fields being transformed; it is also antisymmetric with respect to the interchange of its indices. For the cases of spinor and vector fields, $S^{\nu\sigma}_{rs}$ is given by

$$S^{\nu\sigma}_{rs} = -i\frac{1}{2}\sigma^{\nu\sigma}_{rs} \text{ (spinors)} \quad (3)$$

$$= \delta^\nu_r\delta^\sigma_s - \delta^\nu_s\delta^\sigma_r \text{ (vectors)} \quad (4)$$

In eq 3, $\sigma^{\nu\sigma} = (i/2)[\gamma^\nu, \gamma^\sigma]$, where the γ's are Dirac matrices. The variation of the Lagrangian with respect to δx_ν and $\Delta\phi_r$ is set equal to zero, and a series of algebraic manipulations yields the continuity equation:[16]

$$\partial_\mu g^\mu = 0 \quad (5)$$

where

$$g^\mu = -T^{\mu\nu}\delta x_\nu + \frac{\partial L}{\partial(\partial_\mu\phi_r)}\Delta\phi_r \quad (6)$$

with $T^{\mu\nu}$ being the canonical energy-momentum tensor:[17]

$$T^{uv} = \frac{\partial L}{\partial(\partial_\mu\phi_r)}\partial^v\phi_r - Lg^{uv} \tag{7}$$

A consequence of eq 5 is that the spatial integral of g^0 is a conserved quantity. To see why this is so, write eq 5 as $\vec{\nabla}\cdot\vec{g} = -\partial_0 g^0$ and integrate over all of three-dimensional space. The left-hand side vanishes because the volume integral is converted to a surface integral and it is assumed that $\vec{g}$ falls off sufficiently rapidly at large distances to ensure that the surface integral vanishes. Equation 5 thus yields

$$\frac{d}{dt}\int d^3x\, g^0 = 0 \tag{8}$$

and we see that $\int d^3x\, g^0$ is conserved—it does not vary in time.

Equations 5−7 summarize the version of Noether's theorem that will be used below. The existence of a conserved quantity implies a continuity relation for a tensor whose rank is one higher than that of the conserved quantity. Recall that conservation of a scalar charge implies a continuity relation for a (vector) current: $\partial_v J^v = 0$. Likewise, the conservation of a 4-vector quantity implies a continuity relation for a second rank tensor current. For example, conservation of the 4-momentum $T^{0\sigma}$ implies $\partial_v T^{v\sigma} = 0$. Here the 4-momentum is an example of a Noether charge. And so on for higher rank tensors.

Equation 6 is used to express the conservation of angular momentum. Inserting eqs 1 and 2 into eq 6, with $\mu = 0$, $\delta_v = 0$ (i.e., homogeneous Lorentz transformations), and $v\sigma = ij$ (i.e., rotations, but not boosts) yields

$$g^0 = -T^{0i}\Delta\omega_{ij}x^j + \frac{\partial L}{\partial(\partial_0\phi_r)}\left(\frac{1}{2}\Delta\omega_{ij}S^{ij}_{rs}\phi_s\right) \tag{9}$$

$$= -\frac{1}{2}T^{0i}\Delta\omega_{ij}x^j - \frac{1}{2}T^{0j}\Delta\omega_{ji}x^i + \pi_r\left(\frac{1}{2}\Delta\omega_{ij}S^{ij}_{rs}\phi_s\right) \tag{10}$$

$$= \{(x^iT^{0j} - x^jT^{0i}) + \pi_r S^{ij}_{rs}\phi_s\}\frac{1}{2}\Delta\omega_{ij} \tag{11}$$

The canonical momentum $\pi_r = \partial L/\partial(\partial_0\phi_r)$ has been introduced, and $\Delta\omega_{ij} = -\Delta\omega_{ji}$ has been used. The $^1/_2\Delta\omega_{ij}$ in eq 11 is immaterial as far as conserved quantities are concerned, because the $\Delta\omega_{ij}$'s are independent of one another. They each represent an infinitesimal angular displacement about a Cartesian axis. Thus, the integral of the contents of the large curly bracket is conserved. Namely, it is the *ij* component (*k*-direction) of the total angular momentum of the fields. This includes intrinsic spins. The resulting expression is

$$J^{ij} = \int d^3x\,\{(x^iT^{0j} - x^jT^{0i}) + \pi_r S^{ij}_{rs}\phi_s\} \tag{12}$$

The parenthetic term is of orbital character. This is obvious when the field yields a massive particle upon its quantization. In the case of electromagnetism, orbital angular momentum can be defined in terms of displacement from an origin of the "center" of a wave of finite transverse extent. The other term is the more important one. It is the spin density:

$$S^{0ij} = \pi_r S^{ij}_{rs}\phi_s \tag{13}$$

whose spatial integration yields the spin:

$$S^{ij} = \int d^3x\, S^{0ij} \tag{14}$$

The above approach is a straightforward means of determining intrinsic spins of classical fields. The spins derive directly from the tensor/spinor character of the fields. Once the tensor/spinor nature of the field is set (e.g., scalar, spinor, vector, rank-2 tensor), the spins are automatic, including quantum versions (respectively, 0, $\pm\hbar/2$, $\pm\hbar$, $\pm 2\hbar$). The general version of Noether's theorem is far-reaching, including higher derivatives of the fields, explicit dependence of the Lagrangian on spacetime, and tensor fields of any rank. In the present paper, the Lagrangians depend on only first derivatives of the fields, and they have no explicit spacetime dependence. Only vector and spinor fields are considered.

Photon Spin. The spin of a free electromagnetic field is readily obtained via Noether's theorem. As used here, the term *free electromagnetic field* means that no charges or currents need to be taken into account. The Lagrangian is

$$L_{EM} = -\frac{1}{4}F_{\theta\pi}F^{\theta\pi} \tag{15}$$

The contravariant second-rank antisymmetric field strength tensor: $F^{\theta\pi} = \partial^\theta A^\pi - \partial^\pi A^\theta$ follows the convention in Jackson.[18] In terms of Cartesian components of electric and magnetic fields, $F^{\theta\pi}$ can be expressed as

$$F^{\theta\pi} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{16}$$

The covariant tensor $F_{\theta\pi}$ differs from $F^{\theta\pi}$ only in the sign of $\vec{E}$. In other words, $F_{\theta\pi}$ is the same as the right-hand side of eq 16, but with $\vec{E}$ replaced with $-\vec{E}$. The spin density is evaluated using eq 13, with S^{ij}_{rs} from eq 4, and $\pi_r = -F^{0r}$ (Appendix A). Thus,

$$S^{0ij} = -F^{0r}(\delta^i_r\delta^j_s - \delta^i_s\delta^j_r)A^s \tag{17}$$

$$= -F^{0i}A^j + F^{0j}A^i \tag{18}$$

This shows that S^{0ij} is the *k*-component of $\vec{E} \times \vec{A}$. For example, $S^{012} = -F^{01}A^2 + F^{02}A^1 = E_xA_y - E_yA_x$, which is the *z*-component of $\vec{E} \times \vec{A}$. The spin density is therefore identified: $\vec{S}_{EM} = \vec{E} \times \vec{A}$. Total spin is obtained by integrating $\vec{S}_{EM}$ over the volume of the field. An important point involves the field's edge region. Because the spin density is integrated over the volume of the wave, the field's edge region is not particularly important. Thus, a circularly polarized plane wave propagating in the *z*-direction can be used to describe $\vec{A}$:

$$\vec{A} = A_0(\cos(\omega t - kz)\hat{x} \pm \sin(\omega t - kz)\hat{y}) \tag{19}$$

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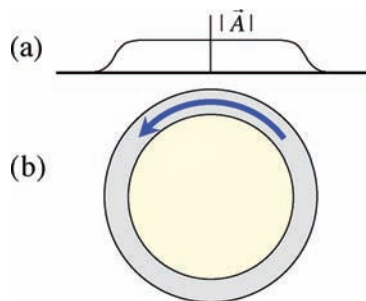*J. Phys. Chem. A, Vol. 113, No. 52, 2009* **15323**

Figure 1. (a) Field strength versus transverse coordinate for a circularly polarized electromagnetic wave, e.g., eq 19 truncated at the wave's edge. (b) The wave's cross section (*xy*-plane), is indicated, showing the constant magnitude region (yellow) and the edge region (gray) where the magnitude falls to zero. In the edge region, the fields have *z*-components. Thus, a momentum current circulates around the periphery (blue arrow).

Using $\vec{E} = -\partial_0 \vec{A}$, with $\vec{A}$ given by eq 19, and $\vec{S}_{EM} = \vec{E} \times \vec{A}$, yields

$$\vec{S}_{EM} = \pm \omega A_0^2 \hat{z} \qquad (20)$$

The electromagnetic energy density $\mathscr{E}_{EM}$ follows just as straightforwardly:

$$\mathscr{E}_{EM} = \frac{1}{2}(E^2 + B^2) = \omega^2 A_0^2 \qquad (21)$$

It is understood that the field is of finite transverse extent. However, because the edge region plays no significant role, eq 19 can be used up to some cutoff distance in the transverse direction. Total spin and energy are obtained by multiplying $\vec{S}_{EM}$ and $\mathscr{E}_{EM}$ by the volume V_0 occupied by the field. Quantization gives the well-known relations $\mathscr{E}_{EM} V_0 = \hbar\omega$ and $\vec{S}_{EM} V_0 = \pm\hbar\hat{z}$.

The above approach differs from the one used by Ohanian,[13] in which spin is assigned to a circulating current at the edge of a field of finite transverse extent (Figure 1).[19] In his paper, the field is taken as circularly polarized with constant magnitude except in its edge region. In expressing the angular momentum density as $\vec{r} \times (\vec{E} \times \vec{B})$, a symmetrized energy momentum tensor

is enlisted, rather than the canonical one given in eq 7. Subsequent (nontrivial) manipulations lead to integration of a density over the wave volume.

As shown above, Noether's theorem yields the result straightaway, and without the assignment of a privileged status to the field's periphery region. Spin density is identified, subsequent calculations are trivial, and interpretation is straightforward. Quantization yields the spin quanta of $\pm\hbar$, identifying the photon as a spin-1 object. The spin of an electromagnetic field is seen to be of a classical nature: quantization serves only to give $\pm\hbar$ eigenvalues.

At the same time, there is no doubt that a circulating flow of momentum exists at the edge of a circularly polarized field of finite transverse extent. This follows from Maxwell's equations. Moreover, the associated angular momentum accounts quantitatively for photon spin. From these unequivocal facts it seems logical to conclude that[13] "This angular momentum is the spin of the wave." However, this overlooks an important point.

The spin of a circularly polarized electromagnetic field is present throughout the field. What happens at the edge is a manifestation of the field's vector nature, but it is not the spin per se. Nonetheless, the question remains, how can these seemingly disparate views—each valid in its own right—be reconciled?

The answer lies with the energy-momentum tensor: canonical versus symmetrized. Ohanian used a symmetric tensor, noting that Belinfante derived a recipe for converting the canonical tensor to a symmetric one,[20] and arguing that the latter is required by gauge invariance and general relativity. The Lorenz gauge, $\partial_\mu A^\mu = 0$, ensures relativistic covariance, whereas a transverse field is useful for quantization. With source-free space there is no issue, and eq 19 fixes the gauge. General relativity (gravity) has nothing to do with the issue at hand. Belinfante needed a symmetric energy-momentum tensor to derive graviton spin. However, gravity is not relevant here.

The canonical tensor reveals intrinsic spin up front, as seen with eq 12. Of course canonical and symmetrized tensors must each yield the correct answer. Differences can only enter in mathematical ardor and ease with which results are interpreted.

Figure 2 illustrates the distinction between the canonical and symmetrized tensor approaches. It is an adaptation of one given by Soper.[21] It is essentially the same as those used to explain the Carnot cycle. Currents are assumed to flow at the edges of
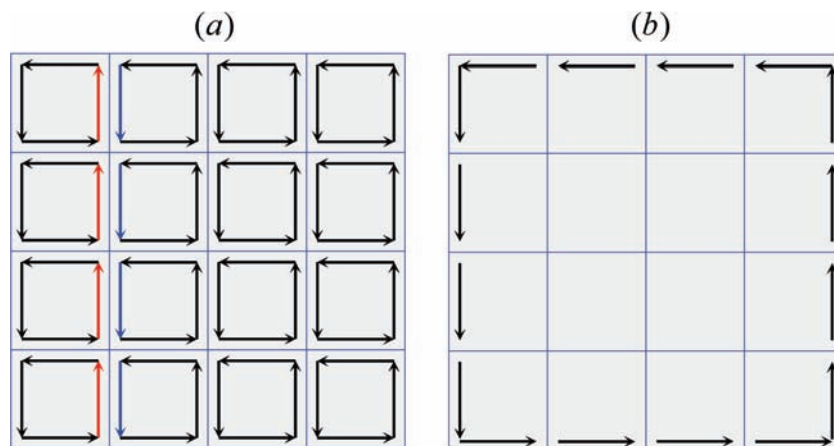

Figure 2. (a) Identical currents flow along the edges of identical squares. Arrows are displaced from the edges for viewing convenience. Each square can be assigned a circulation, i.e., its edge current times its area, that we call its spin. The spin density is uniform throughout. Note: there is no net current in the horizontal or vertical directions for any of the inside edges because of cancellation. For example, red and blue have the same magnitudes but opposite directions. (b) There is no cancellation at the outer edges, so a net current flows on the periphery. As the squares become infinitesimal, the spin density remains constant and the current continues to flow on the periphery.

squares. Each square has an amount of circulation given by its edge current times its area (currents are displaced from edges for viewing convenience). Let us call this circulation the square's spin—an odd spin to be sure, but it serves to make a point.

With small squares, it is appropriate to discuss a spin density, which is the edge current for a given square. Because the spin density in Figure 2 is constant, the total spin of an arbitrary area (which is much larger than the area of an individual square) is the density times this area. When the density is not constant, its integration over a given area yields the spin. Notice that the edge does not enter in a significant way. Thus, identification of the spin density at the outset enables the spin to be calculated trivially. This is akin to the situation in which the canonical tensor yields the spin density, as in eq 12.

Referring to Figure 2a, it is obvious that, except for the periphery, there is no *net* current on any vertical or horizontal line because of cancellation. For example, the magnitude of the current that flows upward (red arrows) is equal to that of the current that flows downward (blue arrows). Cancellation does not occur at the outside edges, so a current circulates at the periphery, as shown in Figure 2b. Though this accounts for the spin, it is the tensor/spinor nature of the field (a vector field in the case of electromagnetism) that dictates the underlying spin density. For example, were the field a scalar (e.g., a Schrödinger field), its spin density would be zero everywhere and the angular momentum tensor would be $x^i T^{0j} - x^j T^{0i}$. A circulation of momentum around the field's edge in this case could not be attributed to a spin density.

To see how this fits with Noether's theorem, consider the symmetrization of the canonical tensor $T^{\mu\nu}$. A new tensor, $\Theta^{\mu\nu}$, is created by adding a term:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\rho\mu\nu} \tag{22}$$

where

$$G^{\rho\mu\nu} = \frac{1}{2}(S^{\rho\mu\nu} - S^{\nu\rho\mu} - S^{\mu\rho\nu}) \tag{23}$$

Because each spin term ($S^{\rho\mu\nu}$, etc.) is antisymmetric with respect to its *last two indices*, $G^{\rho\mu\nu}$ is antisymmetric with respect to its *first two indices*, i.e., $G^{\rho\mu\nu} = -G^{\mu\rho\nu}$. To see how this works, exchange ρ and μ in eq 23. This yields

$$G^{\mu\rho\nu} = \frac{1}{2}(S^{\mu\rho\nu} - S^{\nu\mu\rho} - S^{\rho\mu\nu}) \tag{24}$$

Now replace the second term inside the parentheses (i.e., $-S^{\nu\mu\rho}$) with $S^{\nu\rho\mu}$. This verifies the antisymmetry property: $G^{\rho\mu\nu} = -G^{\mu\rho\nu}$.

In addition, $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ each obey the same continuity equation, i.e., $\partial_\mu\Theta^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu}$. To verify that this is so, use eq 22 to write $\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho G^{\rho\mu\nu}$. Now note that $\partial_\mu\partial_\rho G^{\rho\mu\nu}$ vanishes because $G^{\rho\mu\nu}$ is antisymmetric with respect to its first two indices. Consequently, $\partial_\mu\Theta^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu}$.

Finally, to show that $\Theta^{\mu\nu}$ is symmetric with respect to interchange of its indices, write

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho(G^{\rho\mu\nu} - G^{\rho\nu\mu}) \tag{25}$$

Applying eq 23 yields $G^{\rho\mu\nu} - G^{\rho\nu\mu} = S^{\rho\nu\mu}$. Thus, $\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho S^{\rho\mu\nu}$. The right-hand side is seen to vanish, because $\partial_\rho J^{\rho\mu\nu} = 0 = \partial_\rho(x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu} + S^{\rho\mu\nu}) = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho S^{\rho\mu\nu}$. Thus, the symmetry property: $\Theta^{\mu\nu} = \Theta^{\nu\mu}$ is verified. Belinfante devised this procedure.[20] With the symmetrized tensor, total angular momentum includes spin, but without its explicit identification.

Spin-$^1/_2$. Noether's theorem again yields an expression for the spin density of a classical field, this time the Dirac field. Our concern here is solely with a massive spinor field, so we shall forego inclusion of the electromagnetic field. Thus, the Lagrangian is

$$L_D = \psi^\dagger \gamma^0 (i\gamma^\tau \partial_\tau - m)\psi \tag{26}$$

where the γ^τ are Dirac matrices. Equation 13 is used with S^{ij}_{rs} from eq 3:

$$S^{0ij} = (-i/2)\pi_r \sigma^{ij}_{rs}\phi_s \tag{27}$$

The π_r term is given by $\partial(i\psi^\dagger\gamma^0\gamma^\tau\partial_\tau\psi)/\partial(\partial_0\psi_\alpha) = i\psi^\dagger_\alpha$, where ψ_α is a spinor component. Note: the $\psi^\dagger\gamma^0 m\psi$ term in eq 26 vanishes when L_D is differentiated with respect to $\partial_0\psi_\alpha$. Also, $\phi_r \to \psi_\alpha$ and $\phi_s \to \psi_\beta$; only $\tau = 0$ and the ψ_α spinor component survive differentiation; and $\gamma^0\gamma^0 = 1$. Thus,

$$S^{0ij} = \frac{1}{2}\psi^\dagger_\alpha\sigma^{ij}_{\alpha\beta}\psi_\beta = \frac{1}{2}\psi^\dagger\sigma^{ij}\psi \tag{28}$$

Spin components are obtained by spatial integration of S^{0ij}. The spin density is

$$\vec{S}_D = \frac{1}{2}\psi^\dagger\vec{\sigma}\psi \tag{29}$$

III. Comparison

The results presented in section II are not new, though the facility with which they are obtained using Noether's theorem is noteworthy. As stated earlier, the goal of this paper is an intuitive grasp of intrinsic spin. Photon and electron spins each deserve attention, and they are sufficiently distinct from one another to warrant comparison. It was pointed out that Noether's theorem and canonical energy-momentum tensors yield intrinsic spins of classical fields in a few steps. To further compare canonical and symmetrized tensor approaches, consider the respective expressions for the ij component of the total angular momentum:

$$J^{ij} = \int d^3x \{(x^i T^{0j} - x^j T^{0i}) + S^{0ij}\} \tag{30}$$

$$= \int d^3x \,(x^i\Theta^{0j} - x^j\Theta^{0i}) \tag{31}$$

In eq 30 spin is explicit, while in eq 31 it is not, though it can be made so through manipulation. Specifically, put $\Theta^{0i} = T^{0i} + \partial_k G^{k0i}$ into eq 31:

$$J^{ij} = \int d^3x \{(x^i T^{0j} - x^j T^{0i}) + (x^i\partial_k G^{k0j} - x^j\partial_k G^{k0i})\} \tag{32}$$

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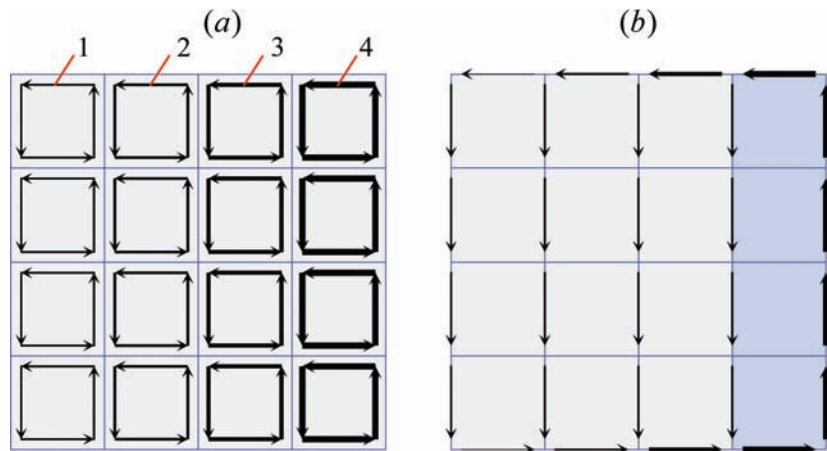*J. Phys. Chem. A, Vol. 113, No. 52, 2009* **15325**

Figure 3. Two ways of assessing total spin. (a) Currents flow along edges of squares. Line thickness indicates currents whose magnitudes range from 1 to 4. Corresponding spins have magnitudes that range from 1 to 4. Summing the spins for the 16 squares gives a total of 40. (b) Net currents along edges are shown. Four superposed rectangles are identified, each having a current of 1 unit. From right to left their areas are 4 (blue shading), 8, 12, and 16 squares, again giving a total spin of 40.

The right parenthetic term, by definition, yields S^{ij} upon integration. To see how this works, integrate the right parenthetic term by parts. For $x^i \partial_k G^{k0j}$ only $k = i$ survives, and for $x^j \partial_k G^{k0i}$ only $k = j$ survives. Thus,

$$S^{ij} = \int d^3x \, (G^{j0i} - G^{i0j}) \qquad (33)$$

Applying eq 23 gives $G^{j0i} - G^{i0j} = S^{0ij}$, showing that eq 33 is $S^{ij} = \int d^3x \, S^{0ij}$. This illustrates the relationship between the canonical and symmetrized tensor approaches. Though each works, the canonical tensor is preferred, as spin appears at the outset.

Spin's presence can be subtle when using $\Theta^{\mu\nu}$. Referring to Figure 2a, summing individual spins gives a total of 16. This is like using eq 14. The same result is obtained using the peripheral current (Figure 2b), which is like using eq 31. Following Soper,[21] the origin is the center, and we consider the contribution from the right edge ($x^1 = 2$). The momentum Θ^{02} is equal to $^1/_2$, so the contribution from the 4 squares that make up this edge is $4x^1 \Theta^{02} = 4$. Because the 4 edges that make up the periphery contribute equally, the total spin of 16 is again obtained. This is easily extended to nonuniform density.[21] Figure 3a shows the magnitude of the spin increasing left to right, but not changing in the vertical direction. Summing individual spins gives a total of 40. The spin is also obtained using net currents (Figure 3b). Horizontal currents are zero due to cancellation, except for the top and bottom, where they decrease in magnitude right to left. Net currents flow on each vertical edge. Four superposed rectangles are identified, each with a current of 1 unit. Their areas are 4 (blue shading), 8, 12, and 16 squares, again giving a total spin of 40.

IV. Discussion and Summary

Classical fields have intrinsic spins that reflect their tensor/spinor nature. Even a scalar field can be said to have an intrinsic spin. It just happens to be zero. In the case of electromagnetism the classical field is a vector field, so it has a spin of 1. When quantized, it has spin components of $\pm\hbar$ along the direction of propagation.

Analogy between photons and classical electromagnetic fields is straightforward. The classical field has a spin density of $\vec{E} \times \vec{A}$, which follows immediately from Noether's theorem and

carries over to the quantum case. Spatially nonuniform fields display momentum currents that are manifestations of the spin density. For example, a momentum current can circulate in the edge region of a circularly polarized electromagnetic field that has constant magnitude everywhere except in the edge region. This is not the spin per se. It is a manifestation of an underlying spin density.

Intrinsic spins of massive particles are subtler. Though their classical fields have spins, there are no real-world analogs of the classical fields of massive particles. A delicate issue is whether the particle or the field is more fundamental. In the standard model of physics, fields play the central role. They contain all of the properties that are passed on to their quanta. In this sense the field is more fundamental. However, as far as an intuitive grasp of spin is concerned, it is not clear that anything is gained by focusing on the massive fields as opposed to their particle counterparts. There is no such quandary with the electromagnetic field. Many photons can be placed in a single mode to create the familiar classical electromagnetic field. This is how lasers work: the presence of photons in a resonator mode encourages the creation of additional photons to favor occupancy of this mode.

The classical Dirac field has a spin density that, like its electromagnetism counterpart, is quadratic in field strength. Spatial variation of the field and therefore the spin density yields a momentum current. This is the same mechanism as in electromagnetism. In each, currents are *manifestations of an underlying spin density*, like the illustrations in Figures 2 and 3.

The spin of the classical Dirac field enters with the spinor representation. In fact, the relationship between intrinsic spin density and its associated momentum flow is a general feature. Namely, spatial variations of the field result in currents that are consequences of the underlying density. We can see how this works by using a vector identity to express the integration of $\vec{S}_D$ as (Appendix B):

$$\int d^3x \, \vec{S}_D = \int d^3x \, \frac{1}{2}(\vec{r} \times (\vec{\nabla} \times \vec{S}_D)) \qquad (34)$$

A simple form for the spinor reveals the mechanism illustrated in Figures 2 and 3. With $\vec{\sigma} = \sigma^{12}\hat{z}$, the nonrelativistic limit of

the spin-up component of the positive energy solution yields: $\bar{S}_D = \frac{1}{2}|\varphi|^2\hat{z}$, and eq 34 becomes

$$\int d^3x\, \bar{S}_D = \int d^3x\, \frac{1}{4}(\vec{r} \times ((\hat{x}\partial_y - \hat{y}\partial_x)|\varphi|^2)) \quad (35)$$

The left-hand side is the usual integration of the spin density. The integrand on the right is recognized as an angular momentum density. For example, with $|\varphi|^2$ taken to be like the distribution in Figure 1, the momentum circulates in the gray annulus, and the integrand points in the $\hat{z}$ direction.

Let us now turn to the spin-$\frac{1}{2}$ *particle* rather than the classical field whose quantization yields it. Being a fermion, it obeys the exclusion rule that defines its statistical property. This fundamental property of the particle is encoded into the theory of quantum mechanics by requiring that the exchange of any two identical fermions results in a sign change of the overall wave function, ψ. In practice, this is achieved by expressing ψ in terms of amplitudes that account for particle exchange:

$$\psi = \phi(x_1,x_2,x_3,...) + c_{12}\phi(x_2,x_1,x_3,...) + ... \quad (36)$$

where $c_{ij} = (-1)^{2s}$, with s the particle spin. Thus, for $s = \frac{1}{2}$ each exchange carries a sign change. In eq 36 the regions between commas (slots) are associated with particles. From left to right: the first slot corresponds to particle 1, the second to particle 2, and so on. For example, $\phi(x_2,x_3,x_1,...)$ is the amplitude for finding particle 1 described by x_2, particle 2 described by x_3, particle 3 described by x_1, and so on. It is understood that the right-hand side of eq 36 includes all possible permutations.

The exchange of two identical massive particles includes a 2π reorientation that is needed to bring the system into registry with the original configuration.[22] An angular displacement of 2π might appear to return a system to its starting point, but this is not the case. An angular displacement of 4π does, whereas 2π does not.[23,24] To accommodate this 2π into a mathematical framework in which the overall wave function changes sign when any two identical fermions are exchanged requires that the particles are endowed with what we call spin-$\frac{1}{2}$. In other words, spin-$\frac{1}{2}$ is the mathematical way of accounting for the fermion property. For massive particles, spin exists in the particle's rest frame.

The efficacy of the mathematics that accounts for spin-$\frac{1}{2}$ is impressive. From the invention of spinors by Cartan in the early part of the twentieth century, through the introduction of spinors to physics by Pauli and Dirac with their matrices (which are, in fact, representations of algebras introduced a half century earlier by Clifford),[25] to the SU(2) and SL(2,$\mathbb{C}$) covering groups,[24] the machinery for *handling* spin-$\frac{1}{2}$ is a "done deal". It obeys the same Lie algebra as does integer angular momentum, and so on. The question of *why* a particle has such a property remains.

The particle perspective can lessen spin's mystique. For example, one sees right away why spin−orbit interaction is relativistic. It is not because of electron spin, but instead the magnetic field experienced by the electron as it circulates around, e.g., a proton. This follows trivially from a Lorentz transformation of the electron−proton Coulomb interaction. Spin itself is not relativistic. It persists unaltered in the nonrelativistic limit of the Dirac equation.

Invoking momentum flow in a classical field whose subsequent quantization creates fermions is not a good mnemonic. The reason is that spin density is first imported through the tensor/spinor nature of the fields, and after it is in place a manifestation of it is recognized.

To summarize, intuition regarding spin inevitably comes back to the property of the fermion particle. It is this property that leads to what we call spin, not the other way around. In other words, the particle's statistical property is manifest as spin in the mathematics. Spin is not some mysterious thing that determines a particle's statistics.

Appendix A

$\partial L_{EM}/\partial(\partial_\mu A_\nu)$ is calculated using $L_{EM} = -\frac{1}{4}F_{\theta\pi}F^{\theta\pi}$, where $F_{\theta\pi} = \partial_\theta A_\pi - \partial_\pi A_\theta$. Indices are lowered: $F^{\theta\pi} = g^{\theta\rho}F_{\rho\varepsilon}g^{\varepsilon\pi}$, and deltas are used to keep track of derivatives:

$$\frac{\partial L_{EM}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}g^{\theta\rho}g^{\varepsilon\pi}\frac{\partial(F_{\theta\pi}F_{\rho\varepsilon})}{\partial(\partial_\mu A_\nu)} =$$
$$-\frac{1}{4}g^{\theta\rho}g^{\varepsilon\pi}\{(\delta_\mu^\theta\delta_\nu^\pi - \delta_\mu^\pi\delta_\nu^\theta)F_{\rho\varepsilon} + (\delta_\mu^\rho\delta_\nu^\varepsilon - \delta_\mu^\varepsilon\delta_\nu^\rho)F_{\theta\pi}\}$$

Applying the delta functions to the metrics gives

$$\frac{\partial L_{EM}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\{(g^{\mu\rho}g^{\varepsilon\nu} - g^{\nu\rho}g^{\varepsilon\mu})F_{\rho\varepsilon} +$$
$$(g^{\theta\mu}g^{\nu\pi} - g^{\theta\nu}g^{\mu\pi})F_{\theta\pi}\}$$

$$= -\frac{1}{4}(F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu}$$

Thus, $\partial L_{EM}/[\partial(\partial_\mu A_\nu)] = -F^{\mu\nu}$, and therefore: $\partial L_{EM}/[\partial(\partial_0 A_r)] = \pi_r = -F^{0r}$.

Appendix B

It is shown that $\int d^3x\, \vec{S}_D = \int d^3x\, \frac{1}{2}(\vec{r} \times (\vec{\nabla} \times \vec{S}_D))$ Hereafter, the subscript D will be understood. The vector identity: $\vec{r} \times (\vec{\nabla} \times \vec{S}) = \vec{\nabla}_S(\vec{r} \cdot \vec{S}) - (\vec{r}\cdot\vec{\nabla})\,\vec{S}$, where $\vec{\nabla}_S$ operates only on the $\vec{S}$ part enables us to write

$$\int d^3x\,(\vec{r} \times (\vec{\nabla} \times \vec{S})) = \int d^3x\,(\vec{\nabla}_S(\vec{r}\cdot\vec{S}) - (\vec{r}\cdot\vec{\nabla})\vec{S})$$

The integrand on the right hand side is

$$x(\vec{\nabla}S_x) + y(\vec{\nabla}S_y) + z(\vec{\nabla}S_z) - (x\partial_x + y\partial_y + z\partial_z)\vec{S} =$$
$$x((\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)S_x) + y((\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)S_y) +$$
$$z((\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)S_z) - (x\partial_x + y\partial_y + z\partial_z)\vec{S}$$

Integration by parts yields $-(S_x\hat{x} + S_y\hat{y} + S_z\hat{z}) - (x\partial_x + y\partial_y + z\partial_z)\,\vec{S}$. The first of these terms is $-\vec{S}$. For the second term, integration by parts gives $3\,\vec{S}$. Thus,

$$\int d^3x\,(\vec{r} \times (\vec{\nabla} \times \vec{S})) = \int d^3x\,2\vec{S}$$

as required.

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References and Notes

(1) Wang, X.; Dong, W.; Xiao, C.; Che, L.; Ren, Z.; Dai, D.; Wang, X.; Casavecchia, P.; Yang, X.; Jiang, B.; Xie, D.; Sun, Z.; Lee, S. Y.; Zhang, D. H.; Werner, H.-J.; Alexander, M. H. *Science* **2009**, *322*, 573.

(2) Balucani, N.; Skouteris, D.; Cartechini, L.; Capozza, G.; Segoloni, E.; Casavecchia, P.; Alexander, M. H.; Capecchi, G.; Werner, H.-J. *Phys. Rev. Lett.* **2003**, *91*, 013201.

(3) Lee, S. H.; Lai, L.-H.; Liu, K.; Chang, H. *J. Chem. Phys.* **1999**, *110*, 8229.

(4) Dong, F.; Lee, S.-H.; Liu, K. *J. Chem. Phys.* **2001**, *115*, 1197.

(5) Alekseyev, A. B.; Lieberman, H.-P.; Wittig, C. *J. Chem. Phys.* **2004**, *121*, 9389.

(6) Underwood, J.; Chastaing, D.; Lee, S.; Wittig, C. *J. Chem. Phys.* **2005**, *123*, 84312.

(7) Aquilanti, V.; Candori, R.; Pirani, F. *J. Chem. Phys.* **1988**, *89*, 6157.

(8) Aquilanti, V.; Luzzatti, E.; Pirani, F.; Volpi, G. G. *J. Chem. Phys.* **1988**, *89*, 6165.

(9) Aquilanti, V.; Candori, R.; Mariani, L.; Pirani, F. *J. Phys. Chem.* **1989**, *93*, 130.

(10) Aquilanti, V.; Liuti, G.; Pirani, F.; Vecchiocattivi, F. *J. Chem. Soc., Faraday Trans.* **1989**, *85*, 955.

(11) Aquilanti, V.; Cappelletti, D.; Pirani, F.; Rusin, L. Y.; Sevryuk, M. B.; Toennies, J. P. *J. Phys. Chem.* **1991**, *95*, 8248.

(12) Aquilanti, V.; Cappelletti, D.; Pirani, F. *J. Chem. Soc., Faraday Trans.* **1993**, *89*, 1467.

(13) Ohanian, H. C. *Am. J. Phys.* **1986**, *54*, 500.

(14) Schwabl, F. *Advanced Quantum Mechanics*; Springer: Berlin, 2005.

(15) Soper, D. E. *Classical Field Theory*; Dover: Mineola, NY, 2008.

(16) Schwabl, F. *Advanced Quantum Mechanics*; Springer: Berlin, 2005; pp 261−273.

(17) Soper, D. E. *Classical Field Theory*; Dover: Mineola, NY, 2008; Chapter 3.

(18) Jackson, J. D. *Classical Electrodynamics*, 3rd ed.; Wiley: Hoboken, NJ, 1999; Chapter 11.

(19) Jackson, J. D. *Classical Electrodynamics* 3rd ed.; Wiley: Hoboken, NJ, 1999; p 350.

(20) Belinfante, F. J. *Physica* **1940**, *7*, 449.

(21) Soper, D. E. *Classical Field Theory*; Dover: Mineola, NY, 2008; Chapter 9.

(22) Wittig, C. *J. Phys. Chem. A* **2009**, *113*, 7244.

(23) Sternberg, S. *Group Theory and Physics*; Cambridge University Press: New York, 1995; Chapter 1.

(24) Penrose R.; Rindler, W. *Spinors and Spacetime*; Cambridge University Press: New York, 1986; Chapter 1.

(25) Doran C.; Lasenby, A. *Geometric Algebra for Physicists*; Cambridge University Press: New York, 2003.